On construction of recursion operator and algebra of symmetries for field and lattice systems *

Maciej Błaszak
Physics Department, A. Mickiewicz University,
Umultowska 85, 61-614 Poznań , Poland
e-mail: blaszakm@main.amu.edu.pl

In the paper, developing the idea of V.Sokolov et all. (J.Math.Phys. 40 (1999) 6473) we construct recursion operators and hereditary algebra of symmetries for many field and lattice systems.

I. INTRODUCTION

In the investigation of nonlinear integrable field and lattice systems in (1+1)-dimensions an important role is played by the algebra of symmetries. For these systems for which additionally exists a so called recursion operator with vanishing Nijenhuis torsion, the algebra of symmetries takes the form of the hereditary algebra (centerless Virasoro algebra). Nevertheless, the construction of the recursion operator of a given integrable system is not an easy task. Several methods were proposed of greater or lower generality, see for example refs. [1], [2], [3], [4]. Perhaps the best known is the one based on the implectic-symplectic factorization of the recursion operator, i.e. on a bi-Hamiltonian property of the system cosidered. But then the problem is shifted to the construction of two Poisson structures which is a complicated problem and on a general level requires advances tools [5]. Moreover, it fails in the case of non-Hamiltonian reductions.

Recently, amasingly simple and general approach to the construction of recursion operators was proposed by Sokolov et al. [6]. The method is based on Lax representation of a given hierarchy of integrable systems and allows a construction of a recursion operators through really elementary calculation.

In the paper presented we develop the idea of Sokolov, applying it to different classes of Lax chains for field and lattice systems. Moreover, we construct from Lax representation in a systematic way conformal symmetries, beeing complementary ingredients in the construction of hereditary algebras.

II. PRELIMINARIES

Let us consider the following scalar Lax operators (together with their admissible reductions):

(i) $L = \partial_x^N + u_{N-2}\partial_x^{N-2} + u_{N-3}\partial_x^{N-3} + ... + u_0,$

(ii) $L = \partial_x^N + u_{N-1}\partial_x^{N-1} + u_{N-2}\partial_x^{N-2} + ... + u_0 + \partial_x^{-1}u_{-1},$

(iii) $L = u_N^N\partial_x^N + u_{N-1}\partial_x^{N-1} + u_{N-2}\partial_x^{N-2} + ... + u_0 + \partial_x^{-1}u_{-1} + \partial_x^{-2}u_{-2},$

for field systems and

(iv) $L = \mathcal{E}^{N+\alpha} + u_{N+\alpha-1}\mathcal{E}^{N+\alpha-1} + ... + u_0 + u_{-1}\mathcal{E}^{-1} + ... + u_\alpha\mathcal{E}^\alpha,$

for lattice systems, where $-N < \alpha \leq -1, \;\; N+\alpha \geq 1.$
Here and further on we use the following notation for differential and shift operators

$$\partial_x a(x) = a_x + a\partial_x, \quad D_x a(x) = a_x,$$

$$\mathcal{E}a(n) = a(n+1)\mathcal{E}, \quad Ea(n) = a(n+1).$$

The related Lax equations are

*Supported by KBN research grant no. P03B 113 13

$$(i)\ \ L_{t_q} = \left[L^{\frac{q}{N}}{}_{\geq 0}, L\right],$$

$$(ii)\ \ L_{t_q} = \left[L^{\frac{q}{N}}{}_{\geq 1}, L\right],$$

$$(iii)\ \ L_{t_q} = \left[L^{\frac{q}{N}}{}_{\geq 2}, L\right], \qquad\qquad q = 1, 2, ...$$

$$(iv)\ \ L_{t_q} = \left[L^{\frac{q}{N+\alpha}}{}_{\geq 0}, L\right],$$

where $L^{\frac{q}{N}}$ is a pseudodifferential series of the form $L^{\frac{q}{N}} = \sum_{-\infty}^{q} v_i \partial_x^i$ and $L^{\frac{q}{N}}{}_{\geq r} = \sum_r^q v_i \partial_x^i$. Here v_i are some concrete functions depending on the coefficients of L.

Fixing N (or N and α), each case gives an infinite hierarchy of commuting flows. The case (i) was first considered by Gelfand and Dikii [7], cases $(ii), (iii)$ by Kupershmidt [8] and the case (iv) by Błaszak and Marciniak [9], respectively. For arbitrary flow from $(i) - (iv)$ we are going to construct a respective algebra of symmetries. What we really need for this construction are two invariant objects, i.e. the related conformal (scaling) vector field σ and a recursion operator ϕ

$$\mathcal{L}_K \sigma + \frac{\partial \sigma}{\partial t} = 0, \quad \mathcal{L}_K \phi = 0, \tag{1}$$

where K is a vector field of a given flow and $\mathcal{L}$ means a Lie derivative.

Lemma 1

If

$$\mathcal{L}_\sigma K = \rho K, \quad \mathcal{L}_\sigma \phi = \alpha \phi, \quad \rho, \alpha = const,$$

$$K_n := \phi^n K, \quad \sigma_n := \phi^n \sigma \tag{2}$$

and

$$\mathcal{L}_{\phi\tau} \phi = \phi \mathcal{L}_\tau \phi, \tag{3}$$

for an arbitrary vector field τ, then K_n and σ_n form a hereditary algebra

$$[K_n, K_m] = 0, \ \ [\sigma_n, K_m] = (\rho + \alpha m) K_{n+m}, \ \ [\sigma_n, \sigma_m] = \alpha(m-n)\sigma_{n+m}. \tag{4}$$

The proof the reader can find for example in ref. [5]. Here we show how to extract scaling symmetry and a recursion operator directly from Lax equations.

III. SCALING PROPERTIES OF LAX EQUATIONS

Let us start from scaling symmetries for field systems. Lax equations $(i) - (iii)$ are homogenous with respect to the following scaling

$$\partial_x \to e^\varepsilon \partial_x, \ \ u_{N-i} \to e^{i\varepsilon} u_{N-i}, \ \ \partial_{t_q} \to e^{q\varepsilon} \partial_{t_q}, \ \ L \to e^{N\varepsilon} L. \tag{5}$$

The conformal point transformation takes the form

$$\varphi_\varepsilon \cdot \begin{pmatrix} u_N \\ u_{N-1} \\ \vdots \\ u_0 \\ u_{-1} \\ u_{-2} \end{pmatrix} = \begin{pmatrix} e^{0\varepsilon} u_N(e^\varepsilon x, e^{q\varepsilon} t_q) \\ e^{1\varepsilon} u_{N-1}(e^\varepsilon x, e^{q\varepsilon} t_q) \\ \vdots \\ e^{N\varepsilon} u_0(e^\varepsilon x, e^{q\varepsilon} t_q) \\ e^{(N+1)\varepsilon} u_{-1}(e^\varepsilon x, e^{q\varepsilon} t_q) \\ e^{(N+2)\varepsilon} u_{-2}(e^\varepsilon x, e^{q\varepsilon} t_q) \end{pmatrix}, \tag{6}$$

so the related infinitesimal generator, i.e. conformal symmetry is

$$\sigma := \frac{d}{d\varepsilon}(\varphi_\varepsilon \cdot u)_{|\varepsilon=0} = \begin{pmatrix} 0 \\ u_{N-1} \\ 2u_{N-2} \\ \vdots \\ (N+2)u_{-2} \end{pmatrix} + x \begin{pmatrix} u_N \\ u_{N-1} \\ u_{N-2} \\ \vdots \\ u_{-2} \end{pmatrix}_x + qt_q \begin{pmatrix} u_N \\ u_{N-1} \\ u_{N-2} \\ \vdots \\ u_{-2} \end{pmatrix}_{t_q} . \tag{7}$$

All admissible reductions preserve this scaling property.

Lax representation (iv) for lattice systems is homogenous with respect to the following scaling

$$\mathcal{E} \to e^{\varepsilon}\mathcal{E}, \;\; u_{N+\alpha-i} \to e^{i\varepsilon} u_{N+\alpha-i}, \;\; \partial_{t_q} \to e^{q\varepsilon}\partial_{t_q}, \;\; L \to e^{(N+\alpha)}L. \tag{8}$$

The conformal point transformation and conformal symmetry are

$$\varphi_\varepsilon \cdot \begin{pmatrix} u_{N+\alpha-1} \\ \vdots \\ u_0 \\ \vdots \\ u_\alpha \end{pmatrix} = \begin{pmatrix} e^{1\varepsilon} u_{N+\alpha-1}(n, e^{q\varepsilon}t_q) \\ \vdots \\ e^{(N+\alpha)\varepsilon} u_0(n, e^{q\varepsilon}t_q) \\ \vdots \\ e^{N\varepsilon} u_\alpha(n, e^{q\varepsilon}t_q) \end{pmatrix}, \tag{9}$$

$$\sigma = \begin{pmatrix} u_{N+\alpha-1} \\ 2u_{N+\alpha-2} \\ \vdots \\ (N+\alpha)u_0 \\ \vdots \\ Nu_\alpha \end{pmatrix} + qt_q \begin{pmatrix} u_{N+\alpha-1} \\ u_{N+\alpha-2} \\ \vdots \\ u_0 \\ \vdots \\ u_\alpha \end{pmatrix}_{t_q} .$$

IV. ALGORYTHMIC CONSTRUCTION OF RECURSION OPERATORS

Developing the idea of Sokolov et al. [6] we construct recursion operators directly from Lax hierarchy

$$L_{t_q} = [A_q, L]. \tag{10}$$

The only necessary information is the explicit form of L operator and the ansatz $A_{\overline{q}} = A_q P + R$ relating A operators with different q, without any specific information of the coefficients of A. Here, P is some operator that commutes with L of the form $P = P(L)$ and R is the remainder, hence

$$L_{t_{\overline{q}}} = L_{t_q} P(L) + [R, L]. \tag{11}$$

Remark

In fact, formulas similar to (11) appeared for the first time in $80th$, derived on more general level in the frame of r-matrix formalism (see for example refs. [10], [11], [12], [13]). Nevertheless, they contain r-matrices and involve more calculations. The advantage of presented method is its simplicity, although the roots in r-matrix theory are evident.

The case (i), i.e. the Gelfand-Dikii case, was considered in details in ref. [6], here we concentrate on the cases (ii) and (iii) for field systems and on the lattice scalar Lax equations (iv).

Of course, in order to apply recursion operators, found by the method presented, to the construction of hereditary algebra of symmetries, it is necessary to verify their hereditary property (3).

A. Field systems (ii)

$(N+1)$-field Lax operator and the Lax hierarchy are

$$L = \partial_x^N + u_{N-1}\partial_x^{N-1} + ... + u_0 + \partial_x^{-1}u_{-1},$$

$$L_{t_m} = [A_m, L], \quad A_m = (L^{\frac{m}{N}})_{\geq 1}, \ \ m = 1, 2, ... \ . \tag{12}$$

Let us express the A_{m+N} operator through A_m, L and some remainder R_m

$$\begin{aligned} A_{m+N} &= (L^{\frac{m}{N}}L)_{\geq 1} = (L^{\frac{m}{N}}{}_{\geq 1}L + L^{\frac{m}{N}}{}_{<1}L)_{\geq 1} \\ &= L^{\frac{m}{N}}{}_{\geq 1}L - (L^{\frac{m}{N}}{}_{\geq 1}L)_0 + (L^{\frac{m}{N}}{}_{<1}L)_{\geq 1} \\ &= A_m L + R_m. \end{aligned} \tag{13}$$

Analysing the highest and lowest order terms of R_m we conclude that the remainder is a purely differential operator

$$R_m = a_m + b_m\partial_x + ... + \gamma_m\partial_x^N, \tag{14}$$

hence

$$L_{t_{m+N}} = L_{t_m}L + [R_m, L]. \tag{15}$$

The right hand side of eq. (15) is the pseudodifferential operator: $RHS = v_{2N-1}\partial_x^{2N-1} + ... + v_0 + \partial_x^{-1}v_{-1} + v_{-2}\partial_x^{-1}u_{-1}$. From $v_k = 0$ for $k = 2N-1, ..., N, -2$, we determine R_m coefficients in terms of the coefficients of operators L and L_{t_m}. Comparing the remaining coefficients of both sides of operator equation (15) we get the recurrence formula

$$\begin{pmatrix} u_{N-1} \\ \vdots \\ u_{-1} \end{pmatrix}_{t_{m+N}} = \phi \begin{pmatrix} u_{N-1} \\ \vdots \\ u_{-1} \end{pmatrix}_{t_m} . \tag{16}$$

There are N commuting chains of vector fields generated by the recursion operator ϕ

$$K_{r,n} = \phi^n K_r, \quad r = 1, ..., N, \ \ n = 0, 1, 2, ... \ . \tag{17}$$

Let us pass to the admissible constraints. There are few of them [14], [5]. The first constraint is of the form $u_{-1} = 0$ and is non-Hamiltonian. It means that we cannot apply a bi-Hamiltonian method for construction a recursion operator and so the method presented becomes even more important. The second constraint takes the form $u_{-1} = u_0 = 0$ and preserves the Hamiltonian structure. The last constraint is the so called Kupershmidt reduction [8]

$$L = (-1)^N\partial_x^{-1}L^{\dagger}\partial_x, \ \ q - odd \ \ \Rightarrow \ \ u_{N-1} = 0, \tag{18}$$

where $(a\partial_x^n)^{\dagger} = (-1)^n\partial_x^n a$. Notice that under this constraint half of equations from the hierarchy (10) disappear and only these with odd q remain. Two cases have to be considered. The first one with an even N preserves the recurrence formula (15) as odd m and even N give odd $m+N$. The second case of odd N is more complex as $m+N$ is even and so it is excluded from the hierarchy. In this case we must take

$$\begin{aligned} A_{m+2N} &= (LL^{\frac{m}{N}}L)_{\geq 1} = (LL^{\frac{m}{N}}{}_{\geq 1}L)_{\geq 1} + (LL^{\frac{m}{N}}{}_{<1}L)_{\geq 1} \\ &= LL^{\frac{m}{N}}{}_{\geq 1}L - (LL^{\frac{m}{N}}{}_{\geq 1}L)_0 - (LL^{\frac{m}{N}}{}_{\geq 1}L)_{-1} + (LL^{\frac{m}{N}}{}_{<1}L)_{\geq 1} \\ &= LA_mL + R_m, \end{aligned} \tag{19}$$

where $R_m = a_m\partial_x^{2N-1} + ... + \partial_x^{-1}\gamma_m$. In derivation of the formula (19) we applied the identity $\partial_x^{-1}a_x\partial_x^{-1} = a\partial_x^{-1} - \partial_x^{-1}a$ and the fact that $u_{N-1} = 0$. Hence, the new recurrence formula is

$$L_{t_{m+2N}} = LL_{t_m}L + [R_m, L], \quad\ m - odd. \tag{20}$$

Example 1. Consider Lax operator of the Kaup-Broer system

$$L = \partial_x + u + \partial_x^{-1}v,$$

with the remainder $R_m = a_m\partial_x + b_m$, hence

$$L_{t_{m+1}} = L_{t_m}L + [R_m, L]$$

$$\Updownarrow$$

$$\begin{aligned} u_{t_{m+1}} + \partial_x^{-1}v_{t_{m+1}} = {} & [u_{t_m} - (a_m)_x]\partial_x + [uu_{t_m} + v_{t_m} + a_m u_x - (b_m)_x] \\ & +\partial_x^{-1}[uv_{t_m} - (v_{t_m})_x - v(D_x^{-1}v_{t_m}) + (a_m v)_x] \\ & +[u_{t_m} + (D_x^{-1}v_{t_m}) + b_m]\partial_x^{-1}v. \end{aligned}$$

Finally

$$[u_{t_m} - (a_m)_x] = 0 \;\Rightarrow\; a_m = D_x^{-1}u_{t_m},$$

$$[u_{t_m} + (D_x^{-1}v_{t_m}) + b_m] = 0 \;\Rightarrow\; b_m = -u_{t_m} - D_x^{-1}v_{t_m}$$

and the recursion formula (16) takes the form

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_{m+1}} = \begin{pmatrix} D_x + D_x u D_x^{-1} & 2 \\ v + D_x v D_x^{-1} & -D_x + u \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_{t_m}.$$

Imposing the constraint $v = 0$ we get the Burgers hierarchy with

$$L = \partial_x + u, \quad a_m \to D_x^{-1}u_{t_m}, \;\; b_m \to -u_{t_m}$$

and the recursion operator of the form

$$\phi = D_x + u + u_x D_x^{-1} = D_x + D_x u D_x^{-1}.$$

The Kupershmidt reduction with odd $N = 1$ gives the nonstandard KdV Lax representation

$$L = \partial_x + \partial_x^{-1}v$$

with Lax recursion

$$L_{t_{m+2}} = LL_{t_m}L + [R_m, L], \qquad m - odd,$$

where

$$R_m = a_m\partial_x + b_m + \partial_x^{-1}c_m,$$

$$a_m = D_x^{-1}v_{t_m}, \;\; b_m = 0, \;\; c_m = -D_x v_{t_m} - vD_x^{-1}v_{t_m}$$

and related vector field recursion

$$v_{t_{m+2}} = (D_x^2 + 4v + 2v_x D_x^{-1})v_{t_m}.$$

Example 2. Consider a three field Lax operator

$$L = \partial_x^2 + u\partial_x + v + \partial_x^{-1}w.$$

The appropriate remainder takes the form

$$R_m = a_m\partial_x^2 + b_m\partial_x + c_m,$$

where

$$a_m = \frac{1}{2}D_x^{-1}u_{t_m}, \;\; b_m = \frac{1}{2}uD_x^{-1}u_{t_m} - \frac{1}{4}D_x^{-1}uu_{t_m} + \frac{1}{2}D_x^{-1}v_{t_m},$$

$$c_m = -v_{t_m} - D_x^{-1} w_{t_m}.$$

So the recurrence formula (16) is

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_{m+2}} = \begin{pmatrix} A & \frac{3}{2}D_x + \frac{1}{2}D_x u D_x^{-1} & 3 \\ B & C & 2u \\ D & \frac{3}{2}w + \frac{1}{2}w_x D_x^{-1} & D_x^2 - D_x u + v \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_m},$$

where

$$\begin{aligned}
A &= \frac{1}{4}D_x^2 + D_x v D_x^{-1} - \frac{1}{4}D_x u D_x^{-1} u, \\
B &= \frac{3}{4}v_x + \frac{2}{3}w + (w_x + \frac{1}{2}v_{xx} + \frac{1}{2}u v_x) D_x^{-1} - \frac{1}{4}v_x D_x^{-1} u, \\
C &= D_x^2 + u D_x + v + \frac{1}{2}v_x D_x^{-1}, \\
D &= -\frac{3}{4}w D_x + \frac{1}{4}uw - \frac{5}{4}w_x + \frac{1}{2}[(uw)_x - w_{xx}] D_x^{-1} - \frac{1}{4}w_x D_x^{-1} u
\end{aligned}$$

and hence two commuting hierarchies of vector fields are

$$K_{1,n} = \phi^n K_1, \quad K_1 = \begin{pmatrix} u_x \\ v_x \\ w_x \end{pmatrix},$$

$$K_{2,n} = \phi^n K_2, \quad K_2 = \begin{pmatrix} 2v_x \\ v_{xx} + 2w_x + u v_x \\ -w_{xx} + (uw)_x \end{pmatrix}.$$

The first reduction $w = 0$ leads to $L = \partial_x^2 + u\partial_x + v$, $c_m = -v_{t_m}$ (a_m, b_m are unchanged),

$$\phi = \begin{pmatrix} \frac{1}{4}D_x^2 + D_x v D_x^{-1} - \frac{1}{4}D_x u D_x^{-1} u & \frac{3}{2}D_x + \frac{1}{2}D_x u D_x^{-1} \\ \frac{3}{4}v_x + (\frac{1}{2}v_{xx} + \frac{1}{2}u v_x) D_x^{-1} - \frac{1}{4}v_x D_x^{-1} u & D_x^2 + u D_x + v + \frac{1}{2}v_x D_x^{-1} \end{pmatrix},$$

$$K_1 = \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \; K_2 = \begin{pmatrix} 2v_x \\ v_{xx} + u v_x \end{pmatrix}.$$

The second reduction gives the MKdV hierarchy

$$L = \partial_x^2 + u\partial_x,$$

$$a_m = \frac{1}{2}D_x^{-1} u_{t_m}, \; b_m = \frac{1}{2}u D_x^{-1} u_{t_m} - \frac{1}{4}D_x^{-1} u u_{t_m}, \; c_m = 0,$$

$$\phi = \frac{1}{4}D_x^2 - \frac{1}{4}D_x u D_x^{-1} u, \quad K_1 = u_x, \; K_2 = 0.$$

Finally, the Kupershmidt reduction gives another nonstandard KdV case

$$L = \partial_x^2 + v - \frac{1}{2}\partial_x^{-1} v_x,$$

where

$$R_m = a_m \partial_x + b_m, \quad a_m = \frac{1}{2}D_x^{-1} v_{t_m}, \; b_m = -\frac{1}{2}v_{t_m},$$

and

$$\phi = D_x^2 + v + \frac{1}{2}v_x D_x^{-1}.$$

Notice that $L = \phi^\dagger$ and the rescaling $v \to 4v$ gives the KdV case from the previous example.

B. Field systems *(iii)*

$(N+3)$-field Lax operator and the Lax hierarchy are

$$L = u_N^N \partial_x^N + u_{N-1}\partial_x^{N-1} + ... + u_0 + \partial_x^{-1}u_{-1} + \partial_x^{-2}u_{-2},$$

$$L_{t_m} = [A_m, L], \quad A_m = (L^{\frac{m}{N}})_{\geq 2}, \quad m = 1, 2, ... \, .$$

Let us express the A_{m+N} operator through A_m, L and some remainder R_m

$$\begin{aligned} A_{m+N} &= (L^{\frac{m}{N}}L)_{\geq 2} = (L^{\frac{m}{N}}{}_{\geq 2}L + L^{\frac{m}{N}}{}_{<2}L)_{\geq 2} \\ &= L^{\frac{m}{N}}{}_{\geq 2}L - (L^{\frac{m}{N}}{}_{\geq 2}L)_0 - (L^{\frac{m}{N}}{}_{\geq 2}L)_1 + (L^{\frac{m}{N}}{}_{<2}L)_{\geq 2} \\ &= A_m L + R_m. \end{aligned} \tag{21}$$

Analysing the highest and lowest order terms of R_m we conclude that the remainder is again a purely differential operator

$$R_m = a_m\partial_x^{N+1} + b_m\partial_x^N + ... + \gamma_m, \tag{22}$$

hence

$$L_{t_{m+N}} = L_{t_m}L + [R_m, L]. \tag{23}$$

There are several admissible constraints. The first group is the following: $u_{-2} = 0$, $u_{-2} = u_{-1} = 0$, $u_{-2} = u_{-1} = u_0 = 0$, $u_{-2} = u_{-1} = u_0 = u_1 = 0$. The first three are non-Hamiltonian the last one is Hamiltonian. Another constraint is related with the Kupershmidt reduction

$$L = (-1)^N\partial_x^{-2}L^{\dagger}\partial_x^2, \tag{24}$$

which surviews only odd terms from the hierarchy. Again an even N preserves the recurence formula (23) as odd m and even N give odd $m+N$, while for odd N $m+N$ is even and so it is excluded from the hierarchy. In this case we must take

$$\begin{aligned} A_{m+2N} &= (LL^{\frac{m}{N}}L)_{\geq 2} = (LL^{\frac{m}{N}}{}_{\geq 2}L)_{\geq 2} + (LL^{\frac{m}{N}}{}_{<2}L)_{\geq 2} \\ &= LL^{\frac{m}{N}}{}_{\geq 2}L - (LL^{\frac{m}{N}}{}_{\geq 2}L)_1 - ... - (LL^{\frac{m}{N}}{}_{\geq 2}L)_{-2} + (LL^{\frac{m}{N}}{}_{<2}L)_{\geq 2} \\ &= LA_mL + R_m, \end{aligned} \tag{25}$$

where $R_m = a_m\partial_x^{2N+1} + ... + \partial_x^{-2}\gamma_m$. Hence, the new recurence formula is

$$L_{t_{m+2N}} = LL_{t_m}L + [R_m, L], \qquad m - odd. \tag{26}$$

Example 3. Consider 4-field Lax operator

$$L = u\partial_x + v + \partial_x^{-1}w + \partial_x^{-2}z.$$

The remainder and the recursion operator are

$$R_m = a_m\partial_x^2 + b_m\partial_x + c_m,$$

$$a_m = u^2D_x^{-1}u^{-2}u_{t_m}, \quad b_m = 2D_x^{-3}z_{t_m} + D_x^{-2}w_{t_m} - u_{t_m},$$

$$c_m = -D_x^{-2}z_{t_m} - D_x^{-1}w_{t_m} - v_{t_m},$$

$$\phi = \begin{pmatrix} A & u & u_xD_x^{-2} - uD_x^{-1} & 2u_xD_x^{-3} - 2uD_x^{-2} \\ B & v + uD_x & 2u + v_xD_x^{-2} & 2v_xD_x^{-3} + uD_x^{-1} \\ C & w & v - D_xu + w_xD_x^{-2} + wD_x^{-1} & u + 2w_xD_x^{-3} + 2wD_x^{-2} \\ D & z & z_xD_x^{-2} + 2zD_x^{-1} & E \end{pmatrix},$$

$$
\begin{aligned}
&A = v + uD_x + (u_{xx} + 2v_x)u^2 D_x^{-1} u^{-2},\\
&B = 2w + (v_{xx} + 2w_x)u^2 D_x^{-1} u^{-2} + 2wuu_x D_x^{-1} u^{-2},\\
&C = 3z - 3w_x - 2wD_x - 2wu^{-1}u_x + [2(u^2 z)_x - (u^2 w)_{xx}]D_x^{-1} u^{-2},\\
&D = -3z_x - 2zD_x - 2zu^{-1}u_x - (u^2 z)_{xx} D_x^{-1} u^{-2},\\
&E = v - D_x u - wD_x^{-1} + 4zD_x^{-2} + 2z_x D_x^{-3}.
\end{aligned}
$$

The hierarchy takes the form

$$
K_{n+1} = \phi^n K_1, \qquad K_1 = \begin{pmatrix} u^2 u_{xx} + 2u^2 v_x \\ u^2 v_{xx} + 2u(uw)_x \\ -(u^2 w)_{xx} + 2(u^2 z)_x \\ -(u^2 z)_{xx} \end{pmatrix}.
$$

The first constraint $z = 0,\ L = u\partial_x + v + \partial_x^{-1} w$ gives

$$
a_m = u^2 D_x^{-1} u^{-2} u_{t_m}, \ \ b_m = D_x^{-2} w_{t_m} - u_{t_m}, \ \ c_m = -D_x^{-1} w_{t_m} - v_{t_m},
$$

$$
\phi = \begin{pmatrix} A & u & u_x D_x^{-2} - uD_x^{-1} \\ B & v + uD_x & 2u + v_x D_x^{-2} \\ \begin{matrix} -3w_x - 2wD_x - 2wu^{-1}u_x \\ -(u^2 w)_{xx} D_x^{-1} u^{-2} \end{matrix} & w & v - D_x u + w_x D_x^{-2} + wD_x^{-1} \end{pmatrix},
$$

$$
K_{n+1} = \phi^n K_1, \qquad K_1 = \begin{pmatrix} u^2 u_{xx} + 2u^2 v_x \\ u^2 v_{xx} + 2u(uw)_x \\ -(u^2 w)_{xx} \end{pmatrix}.
$$

The second constraint $z = w = 0,\ L = u\partial_x + v$ gives

$$
a_m = u^2 D_x^{-1} u^{-2} u_{t_m}, \ \ b_m = -u_{t_m}, \ \ c_m = -v_{t_m},
$$

$$
\phi = \begin{pmatrix} v + uD_x + (u_{xx} + 2v_x)u^2 D_x^{-1} u^{-2} & u \\ v_{xx} u^2 D_x^{-1} u^{-2} & v + uD_x \end{pmatrix},
$$

$$
K_{n+1} = \phi^n K_1, \qquad K_1 = \begin{pmatrix} u^2 u_{xx} + 2u^2 v_x \\ u^2 v_{xx} \end{pmatrix}.
$$

Finally the third constraint $z = w = v = 0,\ L = u\partial_x$ gives

$$
a_m = u^2 D_x^{-1} u^{-2} u_{t_m}, \ \ b_m = -u_{t_m}, \ \ c_m = 0,
$$

$$
\phi = uD_x + u_{xx} u^2 D_x^{-1} u^{-2},
$$

$$
K_{n+1} = \phi^n K_1, \qquad K_1 = u^2 u_{xx}.
$$

C. Lattice systems (iv)

N-field Lax operator and the Lax hierarchy are

$$
L = \mathcal{E}^{N+\alpha} + u_{N+\alpha-1}\mathcal{E}^{N+\alpha-1} + ... + u_\alpha \mathcal{E}^\alpha, \ \ \alpha = -1, ..., -N, \ N \geq 2,
$$

$$
L_{t_m} = [A_m, L], \ \ A_m = (L^{\frac{m}{N+\alpha}})_{\geq 0}, \ \ m = 1, 2, ... \ .
$$

Let us express the $A_{m+N+\alpha}$ operator through A_m, L and some remainder R_m

$$
\begin{aligned}
A_{m+N+\alpha} &= (L^{\frac{m}{N+\alpha}}L)_{\geq 0} = (L^{\frac{m}{N+\alpha}}{}_{\geq 0}L + L^{\frac{m}{N+\alpha}}{}_{<0}L)_{\geq 0} \\
&= L^{\frac{m}{N+\alpha}}{}_{\geq 0}L - (L^{\frac{m}{N+\alpha}}{}_{\geq 0}L)_{-1} - \dots - (L^{\frac{m}{N+\alpha}}{}_{\geq 0}L)_{\alpha} + (L^{\frac{m}{N+\alpha}}{}_{<0}L)_{\geq 0} \\
&= A_m L + R_m.
\end{aligned} \tag{27}
$$

Analysing the highest and lowest order terms of R_m we conclude that the remainder is a shift operator of the form

$$
R_m = a_m \mathcal{E}^{N+\alpha-1} + \dots + \gamma_m \mathcal{E}^{\alpha}, \tag{28}
$$

hence

$$
L_{t_{m+N+\alpha}} = L_{t_m} L + [R_m, L]. \tag{29}
$$

There are two possible reductions. The first one takes the form

$$
u_\alpha \neq 0,\ u_{\alpha+1} = \dots = u_{N+\alpha-1} = 0 \tag{30}
$$

$$
L_{t_m} = [L^m{}_{\geq 0}, L],\ \ m = (N+1)n,\ \ n = 0, 1, 2, \dots,
$$

and includes all Bogoyavlensky lattices, while the second one is

$$
u_0 = u_{-1} = \dots = u_\alpha = 0,\ \ u_1, \dots, u_{N+\alpha-1} \neq 0 \tag{31}
$$

$$
L_{t_m} = [A_m, L],\ \ \ A_m = (L^{\frac{m}{N}})_{\geq 1},\ \ m = 1, 2, \dots
$$

and is a discrete Gelfand-Dikii analog.

We present the calculations on simplest well known examples of Toda and Voltera lattices.

Example 4. Consider infinite Toda lattice $N = 2$, $\alpha = -1$. Hence, we have

$$
L = \mathcal{E} + p + v\mathcal{E}^{-1},\ \ R_m = a_m + b_m \mathcal{E}^{-1}
$$

and the Lax hierarchy (29)

$$
\begin{aligned}
p_{t_{m+1}} + v_{t_{m+1}}\mathcal{E}^{-1} = {} & [p_{t_m} + a_m - (Ea_m)]\mathcal{E} + [pp_{t_m} + v_{t_m} + b_m - (Eb_m)] \\
& + [vp_{t_m} + (E^{-1}p)v_{t_m} + va_m - v(E^{-1}a_m) + b_m(E^{-1}p) \\
& - pb_m]\mathcal{E}^{-1} + [(E^{-1}v)v_{t_m} + b_m(E^{-1}v) - v(E^{-1}b_m)]\mathcal{E}^{-2}.
\end{aligned}
$$

Introducing the operator $\Delta := E - 1$ and its inverse Δ^{-1} such that $\Delta^{-1}f(n) = \sum_{k=-\infty}^{n-1} f(k)$, we find

$$
p_{t_m} + a_m - (Ea_m) = 0\ \ \Rightarrow\ \ a_m = \Delta^{-1}p_{t_m},
$$

$$
(E^{-1}v)v_{t_m} + b_m(E^{-1}v) - v(E^{-1}b_m) = 0\ \ \Rightarrow\ \ b_m = -v\Delta^{-1}Ev^{-1}v_{t_m}
$$

and then

$$
\begin{aligned}
\phi &= \begin{pmatrix} (E^{-1}p) + v(E^{-1}\Delta p)\Delta^{-1}Ev^{-1} & v(1+E^{-1}) \\ 1 + \Delta v \Delta^{-1}Ev^{-1} & p \end{pmatrix} \\
&= \begin{pmatrix} vE^{-1}\Delta p \Delta^{-1}Ev^{-1} & v(1+E^{-1}) \\ (Ev - vE^{-1})\Delta^{-1}Ev^{-1} & p \end{pmatrix},
\end{aligned}
$$

$$
K_{n+1} = \phi^n K_1, \qquad K_1 = \begin{pmatrix} v(n)[p(n) - p(n-1)] \\ v(n+1) - v(n) \end{pmatrix}.
$$

The constraint $p = 0$ leeds to the infinite Volterra system

$$
L = \mathcal{E} + v\mathcal{E}^{-1},\ \ \ L_{t_{2m}} = [L^{2m}{}_{\geq 0}, L],\ \ m = 1, 2, \dots .
$$

Now we find

$$
\begin{aligned}
A_{2m+2} &= (L^{2m}\,L^2)_{\geq 0} = (L^{2m}{}_{\geq 0}L^2 + L^{2m}{}_{<0}L^2)_{\geq 0}\\
&= L^{2m}{}_{\geq 0}L^2 - (L^{2m}{}_{\geq 0}L^2)_{-2} + (L^{2m}{}_{<0}L^2)_{\geq 0}\\
&= A_{2m}L^2 + R_{2m},
\end{aligned}
$$

where $R_{2m} = a_m + b_m\mathcal{E}^{-2}$, so the recursion chain is

$$L_{t_{2m+2}} = L_{t_{2m}}L^2 + [R_{2m}, L]$$

and hence

$$a_m = \Delta^{-1}v_{t_{2m}}, \;\; b_m = -v(E^{-1}v)\Delta^{-1}Ev^{-1}v_{t_{2m}},$$

$$\phi = v + (E^{-1}v) + vE^{-1} + \Delta v(E^{-1}v)\Delta^{-1}Ev^{-1} = v(1+E^{-1})(EvE - v)\Delta^{-1}v^{-1},$$

$$K_1 = v(n)[v(n+1) - v(n-1)].$$

---

[1] A. S. Fokas and R. L. Anderson: J. Math. Phys. **23** (1982) 1066
[2] M. Antonowicz and A. P. Fordy: Comm. Math. Phys. **124** (1989) 2269
[3] G. Tu: J.Math. Phys. **30** (1989) 330
[4] M. Gurses and A. Karasu: J. Math. Phys. **36** (1995) 3485
[5] M. Błaszak: *Multi-Hamiltonian Theory of Dynamical Systems,* Springer, Berlin 1998
[6] M. Gurses, A. Karasu and V. Sokolov: J. Math. Phys. **40** (1999) 6473
[7] I. M. Gelfand and L.A. Dikii: Funct. Anal. Appl. **10** (1976) 259
[8] B. A. Kupershmidt: Commu. Math. Phys. **99** (1988) 51
[9] M. Błaszak and K. Marciniak: J. Math. Phys. **35** (1994) 4661
[10] A. G. Reymann and M. A. Semenov-Tian-Shansky: Funct. Anal. Appl. **17** (1983) 259
[11] A. G. Reymann and M. A. Semenov-Tian-Shansky: Publ. RIMS, Kyoto Univ. **21** (1985) 1237
[12] V. G. Drinfeld: Soviet Math. Dokl. **27** (1983) 68
[13] W. Oevel: J. Math. Phys. **30** (1989) 1140
[14] B. Konopelchenko and W. Oevel: Publ. RIMS, Kyoto Univ. **29** (1993) 581